2.1 The Logistic Equation

We learn

- what the logistic equation is
- how to solve it
- features of the equation in modeling population growth
- birth and death rates
- the limiting population = carrying capacity

Recall the equation that gives exponential population growth:

$$P'(t) = kP, \quad P(t) = Be^{kt}$$

Here P(t) is the size of a population at time t.

We can interpret this in terms of the

- birth rate $\beta = 0.03$ means 3 babies are born to 100 people in unit time.
- death rate δ = the proportion that die in unit time.

We get an equation

$$\frac{dP}{dt} = \beta P - \delta P = (\beta - \delta) P$$

β, δ might not be constant.

The logistic equation is $\frac{dP}{dt} = kP(M-P)$

$\beta - \delta = k(M-P)$, M is a constant, so is k.

Interpretations of the logistic equation:

1. $\frac{dP}{dt}$ is proportional to $M-P$ as well as P.

There is a limit M to population size. As P approaches M, $\beta - \delta$ becomes small.

2. We model birth rate as $\beta_0 - \beta_1 P$

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta) P = (\beta_0 - \delta) P - \beta_1 P^2$$

$$\beta_1 = k \qquad kM = \beta_0 - \delta.$$

Page 82 question 7.
Separate the variables and use partial fractions to solve the IVP. Sketch the graphs of several solutions. Highlight the indicated one.

dx/dt = 4x (7-x), x(0) = 11 $\frac{dx}{dt} = 4x(7-x)$

$\int \frac{dx}{x(7-x)} dx = \int 4\,dt$, $\int \frac{1}{7}\left(\frac{1}{x}+\frac{1}{7-x}\right)dx = \int 4\,dt$

$\frac{1}{7}\left(\ln x - \ln(7-x)\right) = 4t + C$

$\ln\frac{x}{7-x} = 28t + 7C$

$\frac{x}{7-x} = e^{28t+7C} = De^{28t}$, $D = e^{7C}$

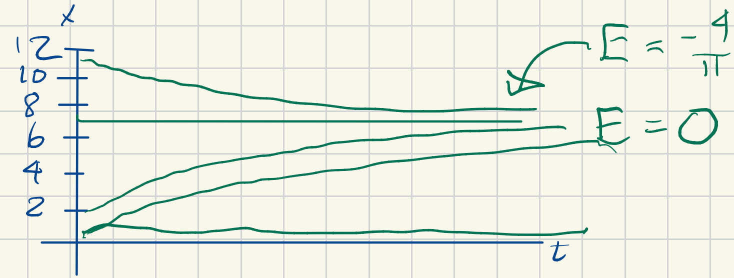
$x = (7-x)De^{28t}$, $x\left(1+D^{28t}\right) = 7De^{28t}$

$x = \frac{7De^{28t}}{1+De^{28t}}$ $x(0) = 11$

$11 = \frac{7D}{1+D}$ $11 + 11D = 7D$, $4D = -11$

$x = \frac{7De^{28t}}{1+De^{28t}} = \frac{7}{Ee^{-28t}+1}$ $E = \frac{-4}{11}$

7 is the carrying capacity.



Partial fractions: we show $\frac{1}{x(7-x)} = \frac{1}{7}\left(\frac{1}{x}+\frac{1}{7-x}\right)$

Put $\frac{1}{x(7-x)} = \frac{A}{x} + \frac{B}{7-x} = \frac{(7-x)A + xB}{x(7-x)}$

$= \frac{7A + (B-A)x}{x(7-x)}$ so $7A = 1$ $B - A = 0$

$B = A = \frac{1}{7}$.

Pre-class Warm-up!!!

Can you remember what the logistic equation is?
Which of the following is it?

a. $dP/dt = kP(M-P)$

b. $dP/dt = (k-M)P$

c. $dP/dt = kP(P-M)$

d. $dP/dt = P^2 (k-M)$

e. None of the above.

Page 82 question 6

Solve dx/dt = 3x (x-5) , x(0) = 2 (or x(0) = 6)

Draw graphs etc.

How can we interpret this equation?

$$\int \frac{dx}{3x(x-5)} = \int \frac{1}{15}\left(\frac{1}{x-5} - \frac{1}{x}\right)dx = \int dt$$

$$\frac{1}{15}\left(\ln(x-5) - \ln(x)\right) = \frac{1}{15}\ln\left(\frac{x-5}{x}\right) = \ln\left(\frac{x-5}{x}\right)^{\frac{1}{15}}$$

$$= t + C$$

$$\left(\frac{x-5}{x}\right)^{\frac{1}{15}} = e^{t+C} = Be^{t} \qquad B = e^{C}$$

$$\frac{x-5}{x} = B^{15}(e^{t})^{15} = De^{15t}$$

$$x - 5 = xDe^{15t}, \quad x(1 - De^{15t}) = 5$$

Partial fractions:

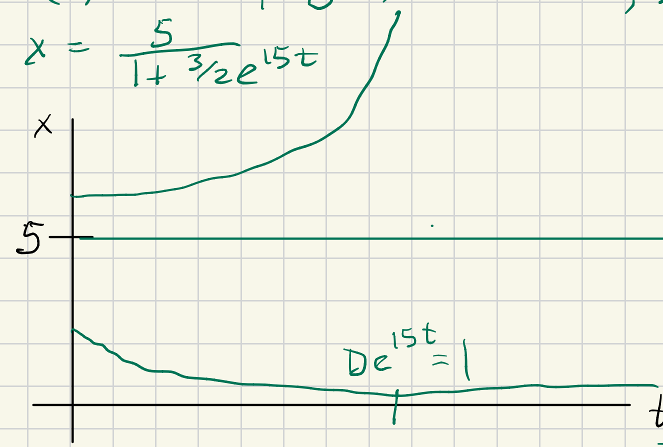
$$\frac{1}{3x(x-5)} = \frac{1}{15}\left(\frac{1}{x-5} - \frac{1}{x}\right)$$

$$x = \frac{5}{1 - De^{15t}}$$

What about the initial values?

$$x(0) = 2 = \frac{5}{1-D}, \quad 2 - 2D = 5, \quad D = -\tfrac{3}{2}$$

$$x = \frac{5}{1 + \tfrac{3}{2}e^{15t}}$$



$$x(0) = 6 = \frac{5}{1-D}, \quad 6 - 6D = 5, \quad D = \tfrac{1}{6} \quad x = \frac{5}{1 - \tfrac{1}{6}e^{15t}}$$

Interpretation:

Doomsday or extinction.

Other birth and death rates: Page 83 question 11

A lake is stocked with fish. The birth rate and death rate are both inversely proportional to $\sqrt{P}$

(a) Show that $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$ for some constant k.

(b) If $P(0) = 100$, and after 6 months there are 169 fish in the lake, how many will there be after 1 year?